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$*$-指数関数から導く切断の具体例

A specific illustration of section derived from $*$-unitary evolution function

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The phase-space formulation of nonrelativistic quantum mechanics is constructed on the basis of a deformation of classical mechanics by using a $*$-product algebra, and it has been illustrated for the MIC-Kepler problem. Its Green's functions are calculated by means of the Moyal product which is one of the $*$-products. In the case where its actual energy E is negative, the Green's function is equal to the infinite series consists of its eigenfunctions which are interpreted as L^2 cross sections of the complex line bundles over $\mathbb{R}^3-\{O\}$.

1 Introduction

The conventional method of calculating Green's function is well-known in operator formalism. A $*$-product algebra counterpart is formulated if one starts with a deformation of the symplectic structures attached to phase space [1][11]. The MIC-Kepler problem is the Hamiltonian system of the hydrogen atom under the influence of the Dirac's magnetic monopole field and the square inverse centrifugal potential force besides the Coulomb's potential force. Iwai and Uwano proved that the MIC-Kepler problem is the 'reduced' Hamiltonian system that comes out of the four-dimensional conformal Kepler problem which is closely related to the four-dimensional harmonic oscillator, if the associated momentum mapping of an S^1 action takes a fixed value μ [6]. It is widely recognized that the three-dimensional hydrogen atom (the quantum-mechanical Kepler problem) has relevance to the four-dimensional harmonic oscillator, and using this Iwai-Uwano's formulation in phase space, the hydrogen atom is the special case when the momentum mapping takes the value zero. Furthermore, they constructed the 'quantised' MIC-Kepler problem by the reduction of the 'quantised' conformal Kepler problem using operator method. Their geometric setting for the reduction process is given by complex line bundles associated with the principal $U(1)\simeq S^1$ bundle $\pi:\dot{\mathbb{R}}^4\to\dot{\mathbb{R}}^3$ where $\dot{\mathbb{R}}^n=\mathbb{R}^n-\{O\}$. From this point of view they obtained the eigenfunctions and Hamiltonian operator of the quantised MIC-Kepler problem [7].
The aim of this paper is to obtain the Green's functions of the MIC-Kepler problem derived from $*$-unitary evolution function which corresponds to unitary operator through the 'Weyl application'. The Weyl application W maps linearly and uniquely a function f on phase space to an operator $W(f)$ in Hilbert space. This approach is carried out on the

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quantum-mechanical Kepler problem, or hydrogen atom [3]. In this way, the $*$-unitary evolution function of four-dimensional oscillator is found firstly. Next, the Green's function of the oscillator is calculated from its $*$-unitary evolution function. After that, the Green's function of the oscillator is reduced to that of the MIC-Kepler problem. The reduction process is originated in path integral method. However, it coincides with Iwai-Uwano's reduction process in which the Hilbert space of square integrable complex-valued functions on $\dot{\mathbb{R}}^4$ is restricted to that of eigenfunctions of the momentum operator associated with the S^1 action on $\dot{\mathbb{R}}^4$.
Section 2 is an outline of the MIC-Kepler problem as reduced system, classical and quantum theories are evolved by Iwai and Uwano [6][7].
In Section 3, restricting its actual energy E within negative levels, the conformal Kepler problem is treated as four-dimensional harmonic oscillator. Then its $*$-unitary evolution function is obtained, which lead to the Green's function of the conformal Kepler problem for negative energy.
In Section 4, the Green's function of the MIC-Kepler problem for negative energy are obtained by reducing that of the conformal Kepler problem. The reduction process requires two local coordinates in practice, hence two local expressions of the Green's function are come out.
In Section 5, Iwai-Uwano's reduction process is carried out for the Green's function of the MIC-Kepler problem. It is demonstrated that their reduction process is the same as that of path integral approach.
The author is grateful to Professor Fujii for presenting his paper [2] which have been of great use in propounding that Green's function may be considered as the cross section of the fibre bundle associated with a principal fibre bundle, or more precisely, as the tensor product of the cross sections.

2 The MIC-Kepler problem as reduced system

This section is a concise explanation of the MIC-Kepler problem in terms of fiber bundle. In 1970, McIntosh and Cisneros studied the dynamical system describing the motion of a charged particle under the magnetic force due to Dirac's monopole field of strength $-\mu$ and the square inverse centrifugal potential force besides the Coulomb's potential force.
The Hamiltonian description for the MIC-Kepler problem is given by Iwai and Uwano as follows. The MIC-Kepler problem is the reduced Hamiltonian system of the 4-dimensional conformal Kepler problem by an S^1 action, if the associated momentum mapping ψ takes a nonzero fixed value μ.
The S^1 action on $\dot{\mathbb{R}}^4$ is defined by a 4×4 matrix $T(\nu)$:

$$\dot{\mathbb{R}}^4 \ni \boldsymbol{u} = (u_1,\, u_2,\, u_3,\, u_4) \longmapsto T(\nu)\boldsymbol{u} \in \dot{\mathbb{R}}^4 \qquad \nu \in [0, 4\pi]$$

where

$$T(\nu) = \begin{pmatrix} R(\nu) & O \\ O & R(\nu) \end{pmatrix} \qquad R(\nu) = \begin{pmatrix} \cos\frac{\nu}{2} & -\sin\frac{\nu}{2} \\ \sin\frac{\nu}{2} & \cos\frac{\nu}{2} \end{pmatrix}.$$

The bundle projection π is

$$\begin{array}{rcl} \pi : \dot{\mathbb{R}}^4 & \longrightarrow & \dot{\mathbb{R}}^3 \\ \boldsymbol{u} & \longmapsto & \boldsymbol{x} = (x,\, y,\, z) \end{array} \qquad \text{where} \quad \begin{cases} x(\boldsymbol{u}) = 2(u_1 u_3 + u_2 u_4) \\ y(\boldsymbol{u}) = 2(-u_1 u_4 + u_2 u_3) \\ z(\boldsymbol{u}) = {u_1}^2 + {u_2}^2 - {u_3}^2 - {u_4}^2 \,. \end{cases}$$

One can easily verify that $u^2 \equiv \sum_{j=1}^{4} {u_j}^2 = \sqrt{x^2+y^2+z^2} \equiv r$ is invariant under the S^1 action.
The S^1 action on $T^*\dot{\mathbb{R}}^4$ is defined by the lift of the one on $\dot{\mathbb{R}}^4$:

$$T^*\dot{\mathbb{R}}^4 \ni (\boldsymbol{u}\,,\, \boldsymbol{\rho}) \longmapsto (T(\nu)\boldsymbol{u}\,,\, T(\nu)\boldsymbol{\rho}) \in T^*\dot{\mathbb{R}}^4 \qquad \nu \in [0, 4\pi]\,.$$

The momentum mapping ψ is

$$\psi(\boldsymbol{u}\,,\, \boldsymbol{\rho}) = \frac{1}{2}(-u_2\rho_1 + u_1\rho_2 - u_4\rho_3 + u_3\rho_4)\,.$$

It is easy to see that ψ is invariant under the S^1 action.
Let $\psi^{-1}(\mu) \subset T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4$ be a subset s.t. $\psi^{-1}(\mu) = \{(\boldsymbol{u}\,,\, \boldsymbol{\rho}) \in T^*_{\boldsymbol{u}}\dot{\mathbb{R}}^4 \,|\, \psi(\boldsymbol{u}\,,\, \boldsymbol{\rho}) = \mu\}$.

The conformal Kepler problem is a triple $(T^*\dot{\mathbb{R}}^4\,,\, d\boldsymbol{\rho}\wedge d\boldsymbol{u}\,,\, H)$ where

$$d\boldsymbol{\rho}\wedge d\boldsymbol{u} \equiv \sum_{j=1}^{4} d\rho_j \wedge du_j \qquad H(\boldsymbol{u}\,,\, \boldsymbol{\rho}) = \frac{1}{2m}\left(\frac{1}{4u^2}\sum_{j=1}^{4} {\rho_j}^2\right) - \frac{k}{u^2}\,,$$

m and k are positive constants for mass of electron (a charged particle) and Coulomb's potential respectively.
The following theorem is established.

Theorem 1 (Iwai-Uwano [6], Theorem 3.1)
The MIC-Kepler problem is the Hamiltonian system $(T^*\dot{\mathbb{R}}^3,\, \sigma_\mu\,,\, H_\mu)$ *s.t.*

$$\begin{cases} H_\mu\,(\boldsymbol{x},\, \boldsymbol{p}) = \dfrac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \dfrac{\mu^2}{2mr^2} - \dfrac{k}{r} \\[2ex] \sigma_\mu = dp_x \wedge dx \,+\, dp_y \wedge dy \,+\, dp_z \wedge dz \,+\, \Omega_\mu \end{cases}$$

where Ω_μ *stands for Dirac's monopole field of strength* $-\mu$

$$\Omega_\mu = \frac{-\mu}{r^3}\,(\, x\, dy\wedge dz \,+\, y\, dz\wedge dx \,+\, z\, dx\wedge dy)\,.$$

Iwai and Uwano construct the quantum system associated with the MIC-Kepler problem $(T^*\dot{\mathbb{R}}^3,\, \sigma_\mu\,,\, H_\mu)$ as follows [7]. The quantised conformal Kepler problem is defined as a pair $(L^2(\dot{\mathbb{R}}^4; 4u^2du),\, \hat{H})$ where $L^2(\dot{\mathbb{R}}^4; 4u^2du)$ is the Hilbert space of square integrable complex-valued functions f on $\dot{\mathbb{R}}^4$ and $\hat{H}$ is the Hamiltonian operator given by

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{1}{4u^2}\sum_{j=1}^{4}\frac{\partial^2}{\partial u_j^2}\right) - \frac{k}{u^2}\,.$$

The quantised conformal Kepler problem is reduced by the S^1 action, the resultant system is considered as the quantum system associated with the MIC-Kepler problem.
In quantum mechanics, the momentum operator associated with the S^1 action is defined by

$$\hat{N} = \frac{i\hbar}{2}\left(-u_2\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial u_2} - u_4\frac{\partial}{\partial u_3} + u_3\frac{\partial}{\partial u_4}\right).$$

To fix a momentum eigenvalue of $\hat{N}$ amounts to a restriction of the Hilbert space $L^2(\mathbb{R}^4; 4u^2du)$ to an eigenspace of $\hat{N}$, this procedure corresponds to fixing the momentum value in the above-mentioned classical case.
Let $U(1) \simeq S^1$ act on $\dot{\mathbb{R}}^4 \times \mathbb{C}$ to the left in the form

$$(\boldsymbol{u}, \zeta) \to (T(\nu)\boldsymbol{u}, \exp(il\nu/2)\zeta) \qquad \boldsymbol{u} \in \dot{\mathbb{R}}^4 \qquad \zeta \in \mathbb{C}$$

where l is an arbitrary integer and ν ranges from 0 to 4π. Then the quotient manifold denoted by $\dot{\mathbb{R}}^4 \times_l \mathbb{C}$ is made into a complex line bundle $L_l = (\dot{\mathbb{R}}^4 \times_l \mathbb{C}, \pi_l, \dot{\mathbb{R}}^3)$, where π_l is the projection, $\pi_l : \dot{\mathbb{R}}^4 \times_l \mathbb{C} \to \dot{\mathbb{R}}^3$. The L_l is called the complex line bundle associated with the $U(1)$ bundle $\pi : \dot{\mathbb{R}}^4 \to \dot{\mathbb{R}}^3$, which were treated to globally describe Dirac's monopole.
If a complex-valued function f satisfies

$$f(T(\nu)\boldsymbol{u}) = \exp(il\nu/2)f(\boldsymbol{u})$$

this f is called ρ_l equivariant, and ρ_l-equivariant functions on $\dot{\mathbb{R}}^4$ are in one-to-one correspondence with cross sections in L_l. Further, ρ_l-equivariant functions f satisfies

$$\hat{N}f = -(l\hbar/2)f\,,$$

thus f turns out to be an eigenfunction of $\hat{N}$ which corresponds uniquely to a cross section in L_l. In this way, the restriction to the ρ_l-equivariant functions and the introduction of the complex line bundle L_l are the geometric consequence of the conservation of the angular momentum associated with the $U(1) \simeq S^1$ action.
The restriction of $L^2(\mathbb{R}^4; 4u^2du)$ to the ρ_l-equivariant functions can be identified with the space of square integrable cross sections in L_l, denoted by Γ_l. The reduced quantum system is defined on Γ_l as the following theorem.

Theorem 2 (Iwai-Uwano [7], Theorem 3.1)
By an S^1 action, the Hilbert space $L^2(\mathbb{R}^4; 4u^2du)$ is reduced to the Hilbert space Γ_l, l being an integer, of square integrable cross sections in the complex line bundles L_l over $\dot{\mathbb{R}}^3$. The L_l is endowed with the linear connection ∇ whose curvature form gives Dirac's monopole field of strength $-l\hbar/2$. If $l = 0$, the L_l becomes a trivial bundle $\dot{\mathbb{R}}^3 \times \mathbb{C}$, and Dirac's monopole field vanishes.

Theorem 3 (Iwai-Uwano [7], Theorem 4.1)
The quantised conformal Kepler problem $(L^2(\mathbb{R}^4; 4u^2du), \hat{H})$ is reduced to the quantum system $(\Gamma_l, \hat{H}_l)$, $\hat{H}_l$ is the Hamiltonian operator given by

$$\hat{H}_l = -\frac{\hbar^2}{2m}\sum_{j=1}^{3}\nabla_j^2 + \frac{(l\hbar/2)^2}{2mr^2} - \frac{k}{r}\,,$$

where ∇_j stands for the covariant derivation of ∂/∂_j with respect to the linear connection. We refer to $(\Gamma_l,\, \hat{H}_l)$ as the quantised MIC-Kepler problem. If $l = 0$, the reduced system becomes the hydrogen atom.

Note. In fact, Iwai and Uwano choose units where $\hbar = 1$ and m is set at unity $(m = 1)$.

3 The Green's function of the conformal Kepler problem

In this section, we calculate the Green's function of the conformal Kepler problem for the purpose of obtaining that of the MIC-Kepler problem in the following section.
Let a real parameter E be the actual energy of the MIC-Kepler problem $(\,T^*\dot{\mathbb{R}}^3,\ \sigma_\mu\,,\ H_\mu\,)$, and let us consider the generalized Hamiltonian $\Phi(\boldsymbol{x},\,\boldsymbol{p})$ defined by [3]

$$\Phi(\boldsymbol{x},\,\boldsymbol{p}) \equiv r(H_\mu - E) = r\left\{\frac{1}{2m}\left({p_x}^2 + {p_y}^2 + {p_z}^2\right) + \frac{\mu^2}{2mr^2} - \frac{k}{r} - E\right\}.$$

Then we have

$$\begin{aligned}\left(\pi_\mu^*\Phi\right)(\boldsymbol{u},\,\boldsymbol{\rho}) =& \frac{1}{8m}({\rho_1}^2 + {\rho_2}^2 + {\rho_3}^2 + {\rho_4}^2) - E(u_1^2 + u_2^2 + u_3^2 + u_4^2) - k \\ =& u^2\left\{\frac{1}{2m}\left(\frac{1}{4u^2}\sum_{j=1}^{4}{\rho_j}^2\right) - \frac{k}{u^2} - E\right\} = u^2(H - E)\end{aligned}$$

where $\quad \pi_\mu : \psi^{-1}(\mu) \longrightarrow T^*\dot{\mathbb{R}}^3$.
Because of $r > 0$ the energy hyper surface $H_\mu = E$ is equivalent to the condition $\Phi(\boldsymbol{x},\,\boldsymbol{p}) = 0$, which is preserved by the equation of motion.
The condition $\left(\pi_\mu^*\Phi\right)(\boldsymbol{u},\,\boldsymbol{\rho}) = 0$ gives

$$\frac{1}{2m}\sum_{j=1}^{4}{\rho_j}^2 - 4E\sum_{j=1}^{4}{u_j}^2 = 4k\,. \tag{1}$$

Then, if $E < 0$, the equation (1) is expressed as

$$\frac{1}{2m}\sum_{j=1}^{4}{\rho_j}^2 + 4|E|\sum_{j=1}^{4}{u_j}^2 = 4k\,. \tag{2}$$

We put $4|E| \equiv \frac{1}{2}m\omega^2$ $(m,\,\omega > 0)$. The equation (2) is

$$\frac{1}{2m}\sum_{j=1}^{4}{\rho_j}^2 + \frac{1}{2}m\omega^2\sum_{j=1}^{4}{u_j}^2 = 4k\,. \tag{3}$$

The left side of (3) is the Hamiltonian of 4-dimensional harmonic oscillator $K(\boldsymbol{u},\,\boldsymbol{\rho})$ where $m\omega^2u^2/2$ is its elastic potential consists of the constant m for mass of pendulum and the

constant ω for angular frequency.
Then, (2) can be considered as 4-dim. harmonic oscillator such that its Hamiltonian (actual energy) equals $4k$ with $m\omega^2/2 = -4E$.
Therefore, we solve this oscillator by means of the Moyal product, especially the $*$-unitary evolution function as follows.

Definition 1 *For a Hamiltonian $H(\boldsymbol{x}, \boldsymbol{p})$ on phase space $(T^*\mathbb{R}^n, d\boldsymbol{p} \wedge d\boldsymbol{x})$ where $d\boldsymbol{p} \wedge d\boldsymbol{x} = \sum_{j=1}^{n} dp_j \wedge dx_j$, and $t \in \mathbb{R}$ the following series $U_*(\boldsymbol{x}, \boldsymbol{p}; t)$ is called $*$-unitary evolution function, or $*$-exponential.*

$$U_*(\boldsymbol{x}, \boldsymbol{p}; t) = 1 + \frac{it}{\hbar}H + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 H * H + \cdots + \frac{1}{N!}\left(\frac{it}{\hbar}\right)^N \overbrace{H * H * \cdots * H}^{N} + \cdots$$

In general, the above power series is not a convergent series. Instead, the following differential equation is considered to define the $*$-exponential.

$$\begin{cases} U_*(\boldsymbol{x}, \boldsymbol{p}; 0) = 1 \\ -i\hbar \dfrac{\partial U_*}{\partial t} = H * U_* = U_* * H \end{cases}$$

This is the definition after the one adopted in [10][11]. Hereafter the notation $e_*^{\frac{it}{\hbar}H(\boldsymbol{x},\boldsymbol{p})}$ is used to stand for a $*$-exponential instead of $U_*(\boldsymbol{x}, \boldsymbol{p}; t)$ throughout the paper, because $e_*^{\frac{it}{\hbar}H(\boldsymbol{x},\boldsymbol{p})}$ expresses the Hamiltonian $H(\boldsymbol{x}, \boldsymbol{p})$.
In order to obtain the $*$-exponential of n-dimensional harmonic oscillator $e_*^{\frac{it}{\hbar}K(\boldsymbol{x},\boldsymbol{p})}$ where

$$K(\boldsymbol{x}, \boldsymbol{p}) = \sum_{j=1}^{n} \frac{1}{2m}p_j^2 + \frac{1}{2}m\omega^2 x_j^2 = \frac{1}{2m}\sum_{j=1}^{n} p_j^2 + \frac{1}{2}m\omega^2 \sum_{j=1}^{n} x_j^2,$$

the following differential equation :

$$\begin{aligned} -i\hbar \frac{\partial}{\partial t} e_*^{\frac{it}{\hbar}K} &= K * e_*^{\frac{it}{\hbar}K} = e_*^{\frac{it}{\hbar}K} * K \\ &= \left(K - \frac{\hbar^2\omega^2}{4} n \frac{\partial}{\partial K} - \frac{\hbar^2\omega^2}{4} K \frac{\partial^2}{\partial K^2}\right) e_*^{\frac{it}{\hbar}K} \end{aligned}$$

with the initial condition $e_*^{\frac{it}{\hbar}K}\Big|_{t=0} = 1$ is solved explicitly.

Proposition 1 *The $*$-exponential of n-dim. harmonic oscillator is given as*
$$e_*^{\frac{it}{\hbar}K(\boldsymbol{x},\boldsymbol{p})} = \left(\cos\frac{\omega t}{2}\right)^{-n} \exp\left\{i\frac{2}{\hbar\omega}K(\boldsymbol{x}, \boldsymbol{p})\tan\frac{\omega t}{2}\right\}, \quad \frac{\omega t}{2} \neq \left(l + \frac{1}{2}\right)\pi, \ {}^{\forall}l \in \mathbb{Z}.$$

The next purpose is to construct the Green's function of 4-dim. harmonic oscillator. Since its $*$-exponential function $e_*^{\frac{it}{\hbar}K}$ has singularities on real axis t ($t \geq 0$), there is an attempt to shift variable from t to $z' \equiv t + iy'$ ($y' \neq 0$) [10]. Then, one can verify the following differential equation :

$$\begin{aligned} -i\hbar \frac{\partial}{\partial z'} e_*^{\frac{iz'}{\hbar}K} &= K * e_*^{\frac{iz'}{\hbar}K} = e_*^{\frac{iz'}{\hbar}K} * K \\ &= \left(K - \frac{\hbar^2\omega^2}{4} n \frac{\partial}{\partial K} - \frac{\hbar^2\omega^2}{4} K \frac{\partial^2}{\partial K^2} \right) e_*^{\frac{iz'}{\hbar}K} \end{aligned}$$

with the initial condition $e_*^{\frac{iz'}{\hbar}K}\Big|_{t=0} = e_*^{-\frac{y'}{\hbar}K}$ gives the following solution.

$$e_*^{\frac{iz'}{\hbar}K(\boldsymbol{x},\boldsymbol{p})} = \left(\cos\frac{\omega z'}{2}\right)^{-n} \exp\left\{ i\frac{2}{\hbar\omega} K(\boldsymbol{x},\, \boldsymbol{p}) \tan\frac{\omega z'}{2} \right\}$$

When $n = 4$, its Hamiltonian on phase space $(T^*\dot{\mathbb{R}}^4,\, d\boldsymbol{\rho} \wedge d\boldsymbol{u})$ is

$$K(\boldsymbol{u},\, \boldsymbol{\rho}) = \frac{1}{2m}\sum_{j=1}^{4} \rho_j^2 + \frac{1}{2}m\omega^2 \sum_{j=1}^{4} u_j^2 \equiv \frac{1}{2m}\rho^2 + \frac{1}{2}m\omega^2 u^2 .$$

Suppose $y' > 0$, the inverse Fourier-transform of the following $*$-exponential is calculated,

$$e_*^{\frac{iz'}{\hbar}K\left(\frac{\boldsymbol{u}_i + \boldsymbol{u}_f}{2},\, \boldsymbol{\rho}\right)} = \left(\cos\frac{\omega z'}{2}\right)^{-4} \exp\left\{ i\frac{2}{\hbar\omega} K\left(\frac{\boldsymbol{u}_i + \boldsymbol{u}_f}{2},\, \boldsymbol{\rho}\right) \tan\frac{\omega z'}{2} \right\}$$

where $\boldsymbol{u}_i$ and $\boldsymbol{u}_f$ denote initial point and final point in $\dot{\mathbb{R}}^4$ respectively.

$$\begin{aligned} &\mathscr{F}^{-1}\left[e_*^{\frac{iz'}{\hbar}K\left(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho}\right)} \right] \\ &= \frac{1}{(2\pi\hbar)^4} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\cos\frac{\omega z'}{2}\right)^{-4} e^{i\frac{2}{\hbar\omega}K\left(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho}\right)\tan\frac{\omega z'}{2}} e^{\frac{i}{\hbar}\boldsymbol{\rho}\cdot(\boldsymbol{u}_i-\boldsymbol{u}_f)} d\boldsymbol{\rho} \\ &= \frac{-m^2\omega^2}{4\pi^2\hbar^2} \frac{1}{\sin^2(\omega z')} \exp\left[-i\frac{m\omega}{2\hbar} \frac{1}{\sin(\omega z')} \left\{ (u_i^2 + u_f^2)\cos(\omega z') - 2\boldsymbol{u}_i \cdot \boldsymbol{u}_f \right\} \right] \\ &\eqqcolon \mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i\, ;\, z') \end{aligned} \tag{4}$$

The Green's function of 4-dim. harmonic oscillator is given by the Laplace transform of $\mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i\, ;\, z')$ and limiting as follows (See Figure 1).

$$\begin{aligned} G(\boldsymbol{u}_f,\, \boldsymbol{u}_i; \epsilon) &= \lim_{\Im(z') \to +0} \frac{i}{\hbar} \int_{\Gamma_{y'}} \mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i; z')\, e^{-\frac{i}{\hbar}(\epsilon - iy')z'} dz' \\ &= \lim_{y' \to +0} \frac{i}{\hbar} \int_0^\infty \mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i; t + iy')\, e^{-\frac{y'+i\epsilon}{\hbar}(t+iy')} dt \\ &= \frac{-im^2\omega^2}{4\pi^2\hbar^3} \lim_{y' \to +0} \int_0^\infty e^{-\frac{i}{\hbar}(\epsilon - iy')(t+iy')} \{\sin(\omega t + i\omega y')\}^{-2} \\ &\quad \times \exp\left[-i\frac{m\omega}{2\hbar} \frac{1}{\sin(\omega t + i\omega y')} \left\{ (u_i^2 + u_f^2)\cos(\omega t + i\omega y') - 2\boldsymbol{u}_i \cdot \boldsymbol{u}_f \right\} \right] dt \end{aligned} \tag{5}$$

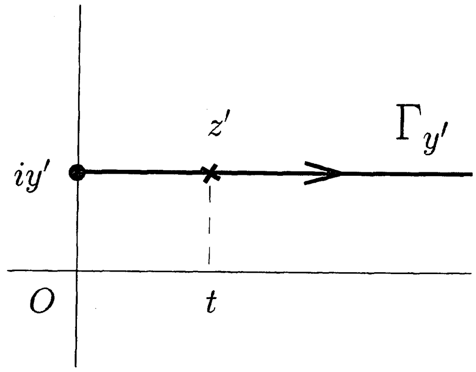

Figure 1: The path of integration $\Gamma_{y'}$ for the Laplace transformation of $\mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i; z')$

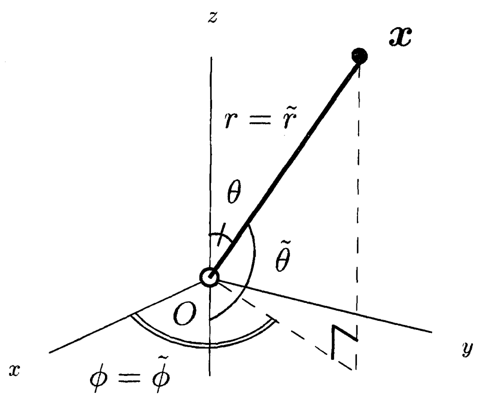

Figure 2: The configuration space $\dot{\mathbb{R}}^3 = \mathbb{R}^3 - \{\mathrm{O}\}$

4 The Green's functions of the MIC-Kepler problem

The Green's function of the MIC-Kepler problem can be obtained by reducing that of the conformal Kepler problem which corresponds to 4-dim. harmonic oscillator if $\epsilon \equiv 4k$ and $m\omega^2 \equiv -8E$.

Assume that E is not on the eigenvalues E_n such that $E_n = \dfrac{-2mk^2}{\hbar^2\,(n+2)^2}$ $\quad (n = 0, 1, 2, \cdots)$.

We consider open subsets of $\dot{\mathbb{R}}^3$ such that $\dot{\mathbb{R}}^3 = U_+ \cup U_-$ where (See Figure 2)

$$\begin{aligned} U_+ &\underset{\leftarrow}{\equiv} \left\{\boldsymbol{x}(r,\, \theta,\, \phi) \in \dot{\mathbb{R}}^3\,;\, r > 0\,,\, 0 \le \theta < \pi\,,\, 0 \le \phi < 2\pi\right\} \\ U_- &\underset{\leftarrow}{\equiv} \left\{\boldsymbol{x}(\tilde{r},\, \tilde{\theta},\, \tilde{\phi}) \in \dot{\mathbb{R}}^3\,;\, \tilde{r} > 0\,,\, 0 \le \tilde{\theta} < \pi\,,\, 0 \le \tilde{\phi} < 2\pi\right\}. \end{aligned}$$

We have two local trivializations, and define two kinds of local coordinate as follows.

$$\tau_+ : \pi^{-1}(U_+) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}),\, \varphi_+(\boldsymbol{u})) = (\boldsymbol{x}(r,\, \theta,\, \phi),\, \exp(i\nu/2)) \in U_+ \times S^1$$

$$\left\{\begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array}\right. ,\quad \left\{\begin{array}{l} u_1 = \sqrt{r}\cos\dfrac{\theta}{2}\cos\dfrac{\nu+\phi}{2}\,,\quad u_2 = \sqrt{r}\cos\dfrac{\theta}{2}\sin\dfrac{\nu+\phi}{2} \\[2ex] u_3 = \sqrt{r}\sin\dfrac{\theta}{2}\cos\dfrac{\nu-\phi}{2}\,,\quad u_4 = \sqrt{r}\sin\dfrac{\theta}{2}\sin\dfrac{\nu-\phi}{2} \end{array}\right.$$

where $\quad 0 \le \nu < 4\pi$.

$$\tau_- : \pi^{-1}(U_-) \ni \boldsymbol{u} \longmapsto (\pi(\boldsymbol{u}),\, \varphi_-(\boldsymbol{u})) = (\boldsymbol{x}(\tilde{r},\, \tilde{\theta},\, \tilde{\phi}),\, \exp(i\tilde{\nu}/2)) \in U_- \times S^1$$

$$\left\{\begin{array}{l} x = \tilde{r}\sin\tilde{\theta}\cos\tilde{\phi} \\ y = \tilde{r}\sin\tilde{\theta}\sin\tilde{\phi} \\ z = -\tilde{r}\cos\tilde{\theta} \end{array}\right. ,\quad \left\{\begin{array}{l} u_1 = \sqrt{\tilde{r}}\sin\dfrac{\tilde{\theta}}{2}\cos\dfrac{\tilde{\nu}+\tilde{\phi}}{2}\,,\quad u_2 = \sqrt{\tilde{r}}\sin\dfrac{\tilde{\theta}}{2}\sin\dfrac{\tilde{\nu}+\tilde{\phi}}{2} \\[2ex] u_3 = \sqrt{\tilde{r}}\cos\dfrac{\tilde{\theta}}{2}\cos\dfrac{\tilde{\nu}-\tilde{\phi}}{2}\,,\quad u_4 = \sqrt{\tilde{r}}\cos\dfrac{\tilde{\theta}}{2}\sin\dfrac{\tilde{\nu}-\tilde{\phi}}{2} \end{array}\right.$$

where $0 \leq \tilde{\nu} = \nu + 2\phi < 4\pi$.
The transition function $g_{-+} = \tau_- \circ \tau_+^{-1} : U_+ \cap U_- \times S^1 \longrightarrow U_+ \cap U_- \times S^1$
is given explicitly as $U_+ \cap U_- \ni \boldsymbol{x} \longmapsto g_{-+}(\boldsymbol{x}) = e^{i\phi(\boldsymbol{x})} \in S^1$.
We calculate the Green's function of the MIC-Kepler problem with the two coordinates as follows. We denote by $G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$ and $G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E)$ the Green's functions expressed in the local coordinate τ_+ and τ_- respectively. $J_l(v)$ is the Bessel function where $\mathbb{Z} \ni l = 2\mu/\hbar$ (see § 2).

Proposition 2 (i) *When $\boldsymbol{x}_i, \boldsymbol{x}_f \in U_+$, the Green's function of the MIC-Kepler problem is*

$$
\begin{aligned}
&G_+(\boldsymbol{x}_f, \boldsymbol{x}_i; E = -m\omega^2/8) \\
&= u_f^2 \lim_{\chi \to 4\pi - 0} \int_0^{\chi} G(\boldsymbol{u}_f, \boldsymbol{u}_i; 4k) \exp\left(i\, l\, \frac{\nu_i - \nu_f}{2}\right) d\nu_i \\
&= -\frac{i^{l+1} m^2\omega^2}{16\pi\hbar^3} \lim_{y' \to +0} \int_0^{\infty} e^{-\frac{i}{\hbar}(4k - iy')(t + iy')} \operatorname{cosec}^2(\omega t + i\omega y') \\
&\qquad \times \exp\left[-i\frac{m\omega}{2\hbar}(r_i + r_f)\cot(\omega t + i\omega y') - i\, l \cdot \frac{\Theta}{2}\right] \\
&\qquad \times J_l\left(\frac{m\omega}{2\hbar}\sqrt{2\boldsymbol{x}_i \cdot \boldsymbol{x}_f + 2 r_i r_f}\, \operatorname{cosec}(\omega t + i\omega y')\right) dt
\end{aligned}
$$

where $$\frac{\Theta}{2} = \tan^{-1}\left[\frac{\sin\frac{\phi_i - \phi_f}{2}}{\cos\frac{\phi_i - \phi_f}{2}} \cdot \frac{\cos\frac{\theta_i + \theta_f}{2}}{\cos\frac{\theta_i - \theta_f}{2}}\right].$$

(ii) *When $\boldsymbol{x}_i, \boldsymbol{x}_f \in U_-$, then the Green's function is written as*

$$
\begin{aligned}
&G_-(\boldsymbol{x}_f, \boldsymbol{x}_i; E = -m\omega^2/8) \\
&= u_f^2 \lim_{\chi \to 4\pi - 0} \int_0^{\chi} G(\boldsymbol{u}_f, \boldsymbol{u}_i; 4k) \exp\left(i\, l\, \frac{\tilde{\nu}_i - \tilde{\nu}_f}{2}\right) d\tilde{\nu}_i \\
&= -\frac{i^{l+1} m^2\omega^2}{16\pi\hbar^3} \lim_{y' \to +0} \int_0^{\infty} e^{-\frac{i}{\hbar}(4k - iy')(t + iy')} \operatorname{cosec}^2(\omega t + i\omega y') \\
&\qquad \times \exp\left[-i\frac{m\omega}{2\hbar}(\tilde{r}_i + \tilde{r}_f)\cot(\omega t + i\omega y') + i\, l \cdot \frac{\tilde{\Theta}}{2}\right] \\
&\qquad \times J_l\left(\frac{m\omega}{2\hbar}\sqrt{2\boldsymbol{x}_i \cdot \boldsymbol{x}_f + 2 \tilde{r}_i \tilde{r}_f}\, \operatorname{cosec}(\omega t + i\omega y')\right) dt
\end{aligned}
$$

where $$\frac{\tilde{\Theta}}{2} = \tan^{-1}\left[\frac{\sin\frac{\tilde{\phi}_i - \tilde{\phi}_f}{2}}{\cos\frac{\tilde{\phi}_i - \tilde{\phi}_f}{2}} \cdot \frac{2\cos(\tilde{\phi}_i - \tilde{\phi}_f)\cos\frac{\tilde{\theta}_i}{2}\cos\frac{\tilde{\theta}_f}{2} + \cos\frac{\tilde{\theta}_i - \tilde{\theta}_f}{2}}{2\cos(\tilde{\phi}_i - \tilde{\phi}_f)\cos\frac{\tilde{\theta}_i}{2}\cos\frac{\tilde{\theta}_f}{2} - \cos\frac{\tilde{\theta}_i + \tilde{\theta}_f}{2}}\right].$$

(iii) *When $\boldsymbol{x}_i, \boldsymbol{x}_f \in U_+ \cap U_-$, $\tilde{\Theta}$ is also written by $\theta = \pi - \tilde{\theta}$ and $\phi = \tilde{\phi}$ as*

$$\frac{\tilde{\Theta}}{2} = \tan^{-1}\left[\frac{\sin\frac{\phi_i - \phi_f}{2}}{\cos\frac{\phi_i - \phi_f}{2}} \cdot \frac{2\cos(\phi_i - \phi_f)\sin\frac{\theta_i}{2}\sin\frac{\theta_f}{2} + \cos\frac{\theta_i - \theta_f}{2}}{2\cos(\phi_i - \phi_f)\sin\frac{\theta_i}{2}\sin\frac{\theta_f}{2} + \cos\frac{\theta_i + \theta_f}{2}}\right].$$

5 Green's function as a series of cross sections

According to the operator formalism, Green's function is written by an infinite series consists of the tensor product of its eigenfunctions. In this section, we show that the equation (5) can be considered as a series consists of the eigenfunctions of 4-dim. harmonic oscillator which is related to the conformal Kepler problem. Since the negative-energy eigenfunctions for the quantised MIC-Kepler problem $(\Gamma_l,\ \hat{H}_l)$ can be obtained from those for the quantised conformal Kepler problem $(L^2(\mathbb{R}^4; 4u^2du),\ \hat{H})$ by the reduction (see § 2), we are able to find the Green's function of the MIC-Kepler problem in a series of its eigenfunctions. In the end, we also come to find the reduction process executed in § 4 is the same as restricting $L^2(\mathbb{R}^4; 4u^2du)$ to the ρ_l-equivariant functions, or cross sections of the complex line bundles.

We consider the Fourier series expansion of (4) on t with a fixed $y' > 0$, where $T \equiv 2\pi/\omega$.

$$\mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i\,; z') = \mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i\,; t+iy') = \sum_{n=-\infty}^{\infty} C_n\, e^{in(2\pi/T)t} = \sum_{n=-\infty}^{\infty} C_n\, e^{in\omega t} \tag{6}$$

$$\begin{aligned} C_n &\equiv \frac{1}{T}\int_{-T/2}^{T/2} \mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i\,; \tau+iy')\, e^{-in(2\pi/T)\tau}\, d\tau = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} \mathscr{K}(\boldsymbol{u}_f,\, \boldsymbol{u}_i\,; \tau+iy')\, e^{-in\omega\tau}\, d\tau \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{K}\left(\boldsymbol{u}_f,\, \boldsymbol{u}_i\,; \frac{t'}{\omega}+iy'\right) e^{-int'}\, dt' \quad (t' \equiv \omega\tau) \end{aligned}$$

We reconstruct the Green's function of 4-dim. harmonic oscillator in the same way as executed on (5) as follows. First, the Laplace transform of (6) gives

$$\frac{i}{\hbar}\int_0^{\infty}\left(\sum_{n=-\infty}^{\infty} C_n\, e^{in\omega t}\right) e^{-\frac{y'+i\epsilon}{\hbar}(t+iy')} dt = e^{\frac{\epsilon y' - i(y')^2}{\hbar}} \sum_{n=-\infty}^{\infty} \frac{\epsilon - n\omega\hbar + iy'}{(\epsilon - n\omega\hbar)^2 + (y')^2}\, C_n\,. \tag{7}$$

Next,

$$\begin{aligned} C_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{K}\left(\boldsymbol{u}_f,\, \boldsymbol{u}_i\,; \frac{t'}{\omega}+iy'\right) e^{-int'}\, dt' \\ &= \frac{-m^2\omega^2}{8\pi^3\hbar^2}\cdot \\ &\quad \int_{-\pi}^{\pi} \frac{e^{-int'}}{\sin^2(t'+i\omega y')} \exp\left[-i\frac{m\omega}{2\hbar}\cdot\frac{(u_i^2+u_f^2)\cos(t'+i\omega y') - 2\boldsymbol{u}_i\cdot\boldsymbol{u}_f}{\sin(t'+i\omega y')}\right] dt'\,. \end{aligned} \tag{8}$$

We put $z' \equiv e^{it'}$, $a \equiv e^{-\omega y'}$ and $v \equiv \dfrac{(az'-1)^2}{a^2(z')^2-1}$, the integration of (8) is

$$\begin{aligned} &\frac{4}{i}\int_{|z'|=1} \frac{(z')^{-n+1}\, e^{-2\omega y'}}{2(z')^2\, e^{-2\omega y'} - (z')^4\, e^{-4\omega y'} - 1} \\ &\qquad \times \exp\left[\frac{m\omega}{2\hbar}\left\{(u_i^2+u_f^2)\frac{(z')^2\, e^{-2\omega y'}+1}{(z')^2\, e^{-2\omega y'}-1} - 4\boldsymbol{u}_i\cdot\boldsymbol{u}_f\frac{z'\, e^{-\omega y'}}{(z')^2\, e^{-2\omega y'}-1}\right\}\right] dz' \\ &= \frac{i}{2}\cdot \\ &\int_{\gamma_{y'}} a^n v^{-2}(1-v)^{1+n}(1+v)^{1-n} \exp\left[\frac{m\omega}{2\hbar}\left\{(u_i^2+u_f^2)\frac{v^2+1}{2v} - \boldsymbol{u}_i\cdot\boldsymbol{u}_f\frac{1-v^2}{v}\right\}\right] dv\,. \end{aligned} \tag{9}$$

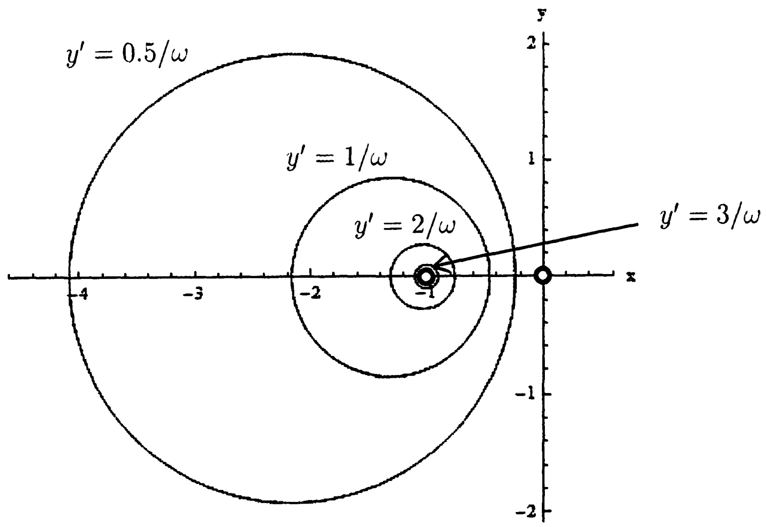

Figure 3: The paths of integration $\gamma_{y'}$ ($y' > 0$)

For every $y' > 0$, the path of integration $\gamma_{y'}$ are anticlockwise circuits around the point $v = -1$ (the points $v = 0$ and $v = 1$ are exterior to any $\gamma_{y'}$, see Figure 3). Therefore, if $1 - n \in \mathbb{N} \cup \{0\}$, the integration of (9) is 0 by Cauchy's theorem. On the other hand, when $1 - n = -1, -2, -3, \cdots$, we change n into $N \equiv n - 2$ ($N = 0, 1, 2, \cdots$).
Then

$$(7) = e^{\frac{\epsilon y' - i(y')^2}{\hbar}} \sum_{N=0}^{\infty} \frac{\epsilon - (N+2)\omega\hbar + iy'}{(\epsilon - (N+2)\omega\hbar)^2 + (y')^2} C_N , \tag{10}$$

$$\begin{aligned} C_N =& \frac{-m^2\omega^2}{8\pi^3\hbar^2} \cdot \frac{i}{2} \cdot \\ & \int_{\gamma_{y'}} a^{N+2} v^{-2} (1-v)^{N+3} (1+v)^{-1-N} \exp\left[\frac{m\omega}{2\hbar}\left\{(u_i^2 + u_f^2)\frac{v^2+1}{2v} - \boldsymbol{u}_i \cdot \boldsymbol{u}_f \frac{1-v^2}{v}\right\}\right] dv \\ =& \frac{-m^2\omega^2}{8\pi^3\hbar^2} \cdot \frac{i}{2} \int_{\gamma_{y'}} 16\, a^{N+2} (1-v)^{N-1} (1+v)^{-1-N} \\ & \times \frac{(1-v)^4}{16v^2} \exp\left[\frac{m\omega}{2\hbar}\left\{(u_i^2 + u_f^2)\frac{v^2+1}{2v} - \boldsymbol{u}_i \cdot \boldsymbol{u}_f \frac{1-v^2}{v}\right\}\right] dv . \end{aligned} \tag{11}$$

Lemma 1

$$\begin{aligned} & \frac{(1-v)^4}{16v^2} \exp\left[\frac{m\omega}{2\hbar}\left\{(u_i^2 + u_f^2)\frac{v^2+1}{2v} - \boldsymbol{u}_i \cdot \boldsymbol{u}_f \frac{1-v^2}{v}\right\}\right] \\ =& \exp\left[-\frac{m\omega}{2\hbar}(u_i^2 + u_f^2)\right] \\ & \sum_{L=0}^{\infty} \sum_{l_1+l_2+l_3+l_4=L} \frac{1}{l_1! l_2! l_3! l_4!} \left\{\frac{1+v}{2(1-v)}\right\}^L H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}} u_1^i\right) H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}} u_1^f\right) H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}} u_2^i\right) \\ & \times H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}} u_2^f\right) H_{l_3}\left(\sqrt{\frac{m\omega}{\hbar}} u_3^i\right) H_{l_3}\left(\sqrt{\frac{m\omega}{\hbar}} u_3^f\right) H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}} u_4^i\right) H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}} u_4^f\right) \end{aligned}$$

where $l_1,\, l_2,\, l_3,\, l_4 \in \mathbb{N} \cup \{0\}$, $\boldsymbol{u}_i = (u^i_1,\, u^i_2,\, u^i_3,\, u^i_4)$, $\boldsymbol{u}_f = (u^f_1,\, u^f_2,\, u^f_3,\, u^f_4)$
$u^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2$, $H_l(X)$ *is the Hermite polynomial.*

Proof. The Mehler's formula is as follows [2]

$$e^{-x^2-y^2} \sum_{n=0}^{\infty} \frac{z^n}{2^n n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp\left[-\frac{1}{1-z^2}(x^2+y^2-2xyz)\right] \tag{12}$$

where $|z| < 1$, $H_n(x)$ is the Hermite polynomial: $H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} e^{-x^2}$.
We consider a product of the left side of (12) for $x_1,\, x_2,\, x_3,\, x_4$, $y_1,\, y_2,\, y_3,\, y_4$ and $z_1 = z_2 = z_3 = z_4 = z$.

$$\left\{e^{-x_1^2-y_1^2} \sum_{l_1=0}^{\infty} \frac{z^{l_1}}{2^{l_1} l_1!} H_{l_1}(x_1) H_{l_1}(y_1)\right\} \left\{e^{-x_2^2-y_2^2} \sum_{l_2=0}^{\infty} \frac{z^{l_2}}{2^{l_2} l_2!} H_{l_2}(x_2) H_{l_2}(y_2)\right\}$$
$$\left\{e^{-x_3^2-y_3^2} \sum_{l_3=0}^{\infty} \frac{z^{l_3}}{2^{l_3} l_3!} H_{l_3}(x_3) H_{l_3}(y_3)\right\} \left\{e^{-x_4^2-y_4^2} \sum_{l_4=0}^{\infty} \frac{z^{l_4}}{2^{l_4} l_4!} H_{l_4}(x_4) H_{l_4}(y_4)\right\}$$

Similarly, the right side of (12) yields

$$\left(\frac{1}{\sqrt{1-z^2}}\right)^4 \exp\left[-\frac{x_1^2+y_1^2-2x_1y_1z}{1-z^2}\right] \exp\left[-\frac{x_2^2+y_2^2-2x_2y_2z}{1-z^2}\right]$$
$$\exp\left[-\frac{x_3^2+y_3^2-2x_3y_3z}{1-z^2}\right] \exp\left[-\frac{x_4^2+y_4^2-2x_4y_4z}{1-z^2}\right]$$

Then, we obtain the following equation.

$$\begin{aligned} &e^{-|\boldsymbol{x}|^2-|\boldsymbol{y}|^2} \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_3=0}^{\infty}\sum_{l_4=0}^{\infty} \frac{z^{l_1+l_2+l_3+l_4}}{2^{l_1+l_2+l_3+l_4} l_1! l_2! l_3! l_4!} H_{l_1}(x_1) H_{l_1}(y_1) \cdots H_{l_4}(x_4) H_{l_4}(y_4) \\ =&\frac{1}{(1-z^2)^2} \exp\left[-\frac{|\boldsymbol{x}|^2+|\boldsymbol{y}|^2-2z\,\boldsymbol{x}\cdot\boldsymbol{y}}{1-z^2}\right] \end{aligned} \tag{13}$$

We put $x_j = \sqrt{\dfrac{m\omega}{\hbar}} u^i_j$ and $y_j = \sqrt{\dfrac{m\omega}{\hbar}} u^f_j$ $(j = 1,2,3,4)$, then (13) is

$$\begin{aligned} &\exp\left[-\frac{m\omega}{\hbar}(u_i^2+u_f^2)\right] \sum_{L=0}^{\infty} \sum_{l_1+l_2+l_3+l_4=L} \frac{1}{l_1! l_2! l_3! l_4!} \left(\frac{z}{2}\right)^L H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}} u^i_1\right) H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}} u^f_1\right) \\ &\qquad H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}} u^i_2\right) H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}} u^f_2\right) \cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}} u^i_4\right) H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}} u^f_4\right) \\ =&\frac{1}{(1-z^2)^2} \exp\left[-\frac{\frac{m\omega}{\hbar}(u_i^2+u_f^2) - \frac{2m\omega}{\hbar} z\, \boldsymbol{u}_i\cdot\boldsymbol{u}_f}{1-z^2}\right]. \end{aligned}$$

We multiply this equation by $\exp\left[\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]$, and find the following one.

$$\exp\left[-\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]\sum_{L=0}^{\infty}\sum_{l_1+l_2+l_3+l_4=L}\frac{1}{l_1!l_2!l_3!l_4!}\left(\frac{z}{2}\right)^L H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^i\right)H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^f\right)$$
$$H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}}u_2^i\right)H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}}u_2^f\right)\cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^i\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^f\right)$$
$$=\frac{1}{(1-z^2)^2}\exp\left[\frac{m\omega}{2\hbar}\left\{(u_i^2+u_f^2)\frac{z^2+1}{z^2-1}-\boldsymbol{u}_i\cdot\boldsymbol{u}_f\frac{4z}{z^2-1}\right\}\right] \tag{14}$$

Since one can verify that $\left|\frac{1+v}{1-v}\right|<1$, let z be $\frac{1+v}{1-v}$. Then we obtain

$$\frac{1}{(1-z^2)^2}=\frac{(1-v)^4}{16v^2},\quad \frac{z^2+1}{z^2-1}=\frac{v^2+1}{2v},\quad \frac{4z}{z^2-1}=\frac{1-v^2}{v}.$$

With these equations and (14), Lemma 1 is deduced.

QED.

Because of Lemma 1, the coefficients C_N $(N=0,1,2\cdots)$ is

$$(11)=\frac{-im^2\omega^2}{\pi^3\hbar^2}\int_{\gamma_{y'}}a^{N+2}(1-v)^{N-1}(1+v)^{-1-N}\exp\left[-\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]$$
$$\sum_{L=0}^{\infty}\sum_{l_1+l_2+l_3+l_4=L}\frac{1}{l_1!l_2!l_3!l_4!}\left\{\frac{1+v}{2(1-v)}\right\}^L H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^i\right)H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^f\right)$$
$$H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}}u_2^i\right)H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}}u_2^f\right)\cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^i\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^f\right)dv$$
$$=\frac{-im^2\omega^2}{\pi^3\hbar^2}\exp\left[-\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]\sum_{L=0}^{\infty}\sum_{l_1+l_2+l_3+l_4=L}\frac{a^{N+2}}{l_1!l_2!l_3!l_4!}\left(\frac{1}{2}\right)^L$$
$$H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^i\right)H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^f\right)\cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^i\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^f\right)$$
$$\int_{\gamma_{y'}}(1-v)^{N-1}(1+v)^{-1-N}\left(\frac{1+v}{1-v}\right)^L dv. \tag{15}$$

We denote by I the integration of (15) and we have

$$I=\int_{\gamma_{y'}}(1-v)^{N-1-L}(1+v)^{-1-N+L}\,dv=\pi i\qquad (L=N).$$

Proof. In the case of $-1-N+L\in\mathbb{N}\cup\{0\}$, the point $v=1$ is exterior to $\gamma_{y'}$ for any $y'>0$, then $I=0$ by Cauchy's theorem. When $-1-N+L=-1,-2,-3,\cdots\Leftrightarrow L\le N$, since $\gamma_{y'}$ make a round of the point $v=-1$ anticlockwise,

$$I=\int_{\gamma_{y'}}\frac{(1-v)^{N-L-1}}{(1+v)^{N-L+1}}dv=\frac{2\pi i}{(N-L)!}\frac{d^{(N-L)}}{dv^{(N-L)}}(1-v)^{N-L-1}\Big|_{v=-1}.$$

The derivative equals 0 when $N-L\in\mathbb{N}$. Then $N-L=0$,

$$I=\int_{\gamma_{y'}}\frac{(1-v)^{-1}}{1+v}\,dv=\frac{2\pi i}{0!}\cdot\frac{1}{1-(-1)}=\pi i\,.$$

QED.

The coefficients C_N $(N=0,1,2\cdots)$ become

$$(15)=\frac{m^2\omega^2}{\pi^2\hbar^2}\exp\left[-\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]\frac{a^{N+2}}{2^N}\sum_{l_1+l_2+l_3+l_4=N}\frac{1}{l_1!l_2!l_3!l_4!}$$
$$H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^i\right)H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^f\right)\cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^i\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^f\right).$$

Therefore, the Laplace transform of (6) is

$$(10)=e^{\frac{\epsilon y'-i(y')^2}{\hbar}}\sum_{N=0}^{\infty}\frac{\epsilon-(N+2)\omega\hbar+iy'}{(\epsilon-(N+2)\omega\hbar)^2+(y')^2}\frac{m^2\omega^2}{\pi^2\hbar^2}\exp\left[-\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]\frac{(e^{-\omega y'})^{N+2}}{2^N}$$
$$\sum_{l_1+l_2+l_3+l_4=N}\frac{1}{l_1!l_2!l_3!l_4!}H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^i\right)H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^f\right)\cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^f\right).\tag{16}$$

The Green's function of 4-dim. harmonic oscillator is reconstructed by the limiting $y'\to +0$ of (16) as follows.

$$\begin{aligned}
&G(\boldsymbol{u}_f,\,\boldsymbol{u}_i;\epsilon)\\
=&\frac{m^2\omega^2}{\pi^2\hbar^2}\exp\left[-\frac{m\omega}{2\hbar}(u_i^2+u_f^2)\right]\sum_{N=0}^{\infty}\frac{1}{\epsilon-(N+2)\hbar\omega}\sum_{l_1+l_2+l_3+l_4=N}\frac{1}{2^Nl_1!l_2!l_3!l_4!}\\
&H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^i\right)H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1^f\right)\cdots H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^i\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4^f\right)\\
=&\sum_{N=0}^{\infty}\frac{1}{\epsilon-(N+2)\hbar\omega}\Psi_N(\boldsymbol{u}_f)\,\Psi_N^*(\boldsymbol{u}_i)
\end{aligned}\tag{17}$$

where $l_1+l_2+l_3+l_4=N$,

$$\begin{aligned}
\Psi_N(\boldsymbol{u})\equiv&\frac{m\omega}{\pi\hbar}\frac{1}{\sqrt{2^Nl_1!l_2!l_3!l_4!}}\exp\left[-\frac{m\omega}{2\hbar}(u_1^2+u_2^2+u_3^2+u_4^2)\right]\\
&H_{l_1}\left(\sqrt{\frac{m\omega}{\hbar}}u_1\right)H_{l_2}\left(\sqrt{\frac{m\omega}{\hbar}}u_2\right)H_{l_3}\left(\sqrt{\frac{m\omega}{\hbar}}u_3\right)H_{l_4}\left(\sqrt{\frac{m\omega}{\hbar}}u_4\right).
\end{aligned}\tag{18}$$

Moreover, $\Psi_N(\boldsymbol{u})$ is written as the following lemma.

Lemma 2

$$\Psi_N(\boldsymbol{u})=\frac{m\omega}{\pi\hbar}\left(\sqrt{\frac{m\omega}{\hbar}}\right)^N\frac{\mathscr{P}(\xi\bar{\xi},\,\eta\bar{\eta})}{\sqrt{k_1!k_2!k_3!k_4!}}\,(\bar{\xi})^{k_1}\,\xi^{k_3}\,(\bar{\eta})^{k_2}\,\eta^{k_4}\exp\left[-\frac{m\omega}{2\hbar}\,(\xi\bar{\xi}+\eta\bar{\eta})\right]$$

where $k_1, k_2, k_3, k_4 \in \mathbb{N} \cup \{0\}$ *s.t.* $k_1 + k_2 + k_3 + k_4 = N$, $\xi = u_1 + iu_2$, $\eta = u_3 + iu_4$ *and* $\mathscr{P}\left(\xi\bar{\xi}\,,\,\eta\bar{\eta}\right)$ *is the following polynomial.*

$$\mathscr{P}\left(\xi\bar{\xi}\,,\,\eta\bar{\eta}\right) = \sum_{j=0}^{k_1}\sum_{s=0}^{k_2} j!s!\left(-\frac{\hbar}{m\omega}\right)^{j+s} {}_{k_1}C_j \cdot {}_{k_3}C_j \cdot {}_{k_2}C_s \cdot {}_{k_4}C_s \left(\xi\bar{\xi}\right)^{-j}\left(\eta\bar{\eta}\right)^{-s}$$

Proof. Iwai and Uwano proved that [7]

$$\begin{aligned}(18) &= \frac{1}{\sqrt{l_1!l_2!l_3!l_4!}}\left(a_1^+\right)^{l_1}\left(a_2^+\right)^{l_2}\left(a_3^+\right)^{l_3}\left(a_4^+\right)^{l_4}\frac{m\omega}{\pi\hbar}e^{-\frac{m\omega}{2\hbar}(u_1^2+u_2^2+u_3^2+u_4^2)}\\ &= \frac{1}{\sqrt{k_1!k_2!k_3!k_4!}}\left(b_1^+\right)^{k_1}\left(b_2^+\right)^{k_2}\left(b_3^+\right)^{k_3}\left(b_4^+\right)^{k_4}\frac{m\omega}{\pi\hbar}e^{-\frac{m\omega}{2\hbar}(u_1^2+u_2^2+u_3^2+u_4^2)} \qquad (19)\end{aligned}$$

where $l_1, l_2, l_3, l_4, k_1, k_2, k_3, k_4 \in \mathbb{N} \cup \{0\}$ s.t. $l_1 + l_2 + l_3 + l_4 = k_1 + k_2 + k_3 + k_4 = N$ and b_j^+ $(j = 1, 2, 3, 4)$ is the linear combination of create operators a_j^+ given as

$$a_j^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}u_j - \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial u_j}\right),\ \left\{\begin{array}{ll} b_1^+ = \frac{1}{\sqrt{2}}(a_1^+ - ia_2^+) \ , & b_2^+ = \frac{1}{\sqrt{2}}(a_3^+ - ia_4^+) \\ b_3^+ = \frac{1}{\sqrt{2}}(a_1^+ + ia_2^+) \ , & b_4^+ = \frac{1}{\sqrt{2}}(a_3^+ + ia_4^+) \,. \end{array}\right.$$

Then

$$\begin{aligned}(19) = \frac{2\alpha}{\pi}\frac{1}{\sqrt{2^{k_1+k_2+k_3+k_4}k_1!k_2!k_3!k_4}} &\left(\sqrt{\alpha}\,\bar{\xi} - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\xi}\right)^{k_1}\left(\sqrt{\alpha}\,\xi - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\bar{\xi}}\right)^{k_3} e^{-\alpha\xi\bar{\xi}}\\ &\left(\sqrt{\alpha}\,\bar{\eta} - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\eta}\right)^{k_2}\left(\sqrt{\alpha}\,\eta - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\bar{\eta}}\right)^{k_4} e^{-\alpha\eta\bar{\eta}} \qquad (20)\end{aligned}$$

where $\xi = u_1 + iu_2$, $\eta = u_3 + iu_4$ s.t. $(\xi,\,\eta) \neq (0,\,0)$ and $\alpha = \dfrac{m\omega}{2\hbar}$.

i)When $\xi = 0$ and $\eta \neq 0$,

$$\left(\sqrt{\alpha}\,\bar{\xi} - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\xi}\right)^{k_1}\left(\sqrt{\alpha}\,\xi - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\bar{\xi}}\right)^{k_3} e^{-\alpha\xi\bar{\xi}} = \left\{\begin{array}{ll} 0 & (k_1,\,k_3) \neq (0,\,0) \\ 1 & (k_1,\,k_3) = (0,\,0)\,. \end{array}\right.$$

ii)When $\xi \neq 0$ and $\eta = 0$,

$$\left(\sqrt{\alpha}\,\bar{\eta} - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\eta}\right)^{k_2}\left(\sqrt{\alpha}\,\eta - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\bar{\eta}}\right)^{k_4} e^{-\alpha\eta\bar{\eta}} = \left\{\begin{array}{ll} 0 & (k_2,\,k_4) \neq (0,\,0) \\ 1 & (k_2,\,k_4) = (0,\,0)\,. \end{array}\right.$$

iii)When $\xi \neq 0$ and $\eta \neq 0$, by induction one can prove easily

$$\left(\sqrt{\alpha}\,\xi - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\bar{\xi}}\right)^{k} e^{-\alpha\xi\bar{\xi}} = \left(2\xi\sqrt{\alpha}\right)^{k} e^{-\alpha\xi\bar{\xi}}\,. \qquad (21)$$

Further, by means of induction calculation with (21), we can show

$$\begin{aligned}&\left(\sqrt{\alpha}\,\bar{\xi} - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\xi}\right)^{k}\left(\sqrt{\alpha}\,\xi - \frac{1}{\sqrt{\alpha}}\frac{\partial}{\partial\bar{\xi}}\right)^{k'} e^{-\alpha\xi\bar{\xi}}\\ &= \left(2\sqrt{\alpha}\right)^{k'} e^{-\alpha\xi\bar{\xi}}\sum_{j=0}^{k} j!(-1)^j 2^{k-j}\left(\sqrt{\alpha}\right)^{k-2j} {}_kC_j \cdot {}_{k'}C_j \left(\bar{\xi}\right)^{k-j}\xi^{k'-j}\,. \qquad (22)\end{aligned}$$

Because of (22), when $\xi\eta \neq 0$, we have

$$
\begin{aligned}
(20) =& \frac{2\alpha}{\pi} \frac{1}{\sqrt{2^{k_1+k_2+k_3+k_4} k_1!k_2!k_3!k_4}} \left(2\sqrt{\alpha}\right)^{k_3+k_4} e^{-\alpha(\xi\bar{\xi}+\eta\bar{\eta})} \\
& \sum_{j=0}^{k_1} j!(-1)^j 2^{k_1-j} \left(\sqrt{\alpha}\right)^{k_1-2j} {}_{k_1}C_j \cdot {}_{k_3}C_j \left(\bar{\xi}\right)^{k_1-j} \xi^{k_3-j} \\
& \sum_{s=0}^{k_2} s!(-1)^s 2^{k_2-s} \left(\sqrt{\alpha}\right)^{k_2-2s} {}_{k_2}C_s \cdot {}_{k_4}C_s \left(\bar{\eta}\right)^{k_2-s} \eta^{k_4-s} \\
=& \frac{m\omega}{\pi\hbar} \frac{1}{\sqrt{k_1!k_2!k_3!k_4}} \left(\sqrt{\frac{m\omega}{\hbar}}\right)^{k_1+k_2+k_3+k_4} \left(\bar{\xi}\right)^{k_1} \xi^{k_3} \left(\bar{\eta}\right)^{k_2} \eta^{k_4} \exp\left[-\frac{m\omega}{2\hbar}\left(\xi\bar{\xi}+\eta\bar{\eta}\right)\right] \\
& \sum_{j=0}^{k_1}\sum_{s=0}^{k_2} j!s! \left(-\frac{\hbar}{m\omega}\right)^{j+s} {}_{k_1}C_j \cdot {}_{k_3}C_j \cdot {}_{k_2}C_s \cdot {}_{k_4}C_s \left(\xi\bar{\xi}\right)^{-j} \left(\eta\bar{\eta}\right)^{-s} . \qquad (23)
\end{aligned}
$$

This equation (23) proves to be true in the case of both i) and ii).

QED.

With Lemma 2, we can express $\Psi_N(\boldsymbol{u})$ by the polar coordinates defined in § 4. First of the two, we use τ_+ and we have

$$
\begin{aligned}
\Psi_N(\boldsymbol{u}) =& \frac{m\omega}{\pi\hbar} \left(\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathscr{P}\left(r\cos^2\frac{\theta}{2}, r\sin^2\frac{\theta}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} \left(\sqrt{r}\cos\frac{\theta}{2}\right)^{k_1+k_3} \left(\sqrt{r}\sin\frac{\theta}{2}\right)^{k_2+k_4} e^{-\frac{m\omega}{2\hbar}r} \\
& \exp\left[-i(k_1-k_2-k_3+k_4)\frac{\phi}{2}\right] \exp\left[-i(k_1+k_2-k_3-k_4)\frac{\nu}{2}\right] .
\end{aligned}
$$

Iwai and Uwano proved that the restriction of $L^2(\mathbb{R}^4; 4u^2du) \ni \Psi_N(\boldsymbol{u})$ to the ρ_l-equivariant functions is equal to the following condition [7]

$$
k_1 + k_2 - k_3 - k_4 = -l \quad (l \in \mathbb{Z}) .
$$

Therefore, the ρ_l-equivariant eigenfunction $\Psi_{N,l}(\boldsymbol{u})$ is given as

$$
\begin{aligned}
\Psi_{N,l}(\boldsymbol{u}) =& \frac{m\omega}{\pi\hbar} \left(\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathscr{P}\left(r\cos^2\frac{\theta}{2}, r\sin^2\frac{\theta}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} \left(\sqrt{r}\cos\frac{\theta}{2}\right)^{k_1+k_3} \left(\sqrt{r}\sin\frac{\theta}{2}\right)^{k_2+k_4} e^{-\frac{m\omega}{2\hbar}r} \\
& \exp\left[-i(k_1-k_2-k_3+k_4)\frac{\phi}{2}\right] \exp\left[i\,l\,\frac{\nu}{2}\right] .
\end{aligned}
$$

Furthermore, since ρ_l-equivariant functions $\Psi_{N,l}(\boldsymbol{u})$ are in one-to-one correspondence with cross sections in L_l, we may introduce $\Psi_N(\boldsymbol{x})$ as the cross sections defined by [9]

$$
\Psi_N(\boldsymbol{x}) \equiv \frac{\sqrt{\pi}}{2} e^{-i\,l\,\nu/2} \Psi_{N,l}(\boldsymbol{u}) . \qquad (24)
$$

We can reconstruct the Green's function of the MIC-Kepler problem with a series of its eigenfunctions given by (24).

$$
\begin{aligned}
&G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E = -m\omega^2/8) \\
&= \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, \Psi_N(\boldsymbol{x}_f)\, \Psi_N^*(\boldsymbol{x}_i) \\
&= \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, e^{-\frac{m\omega}{2\hbar}(r_i + r_f)} \\
&\quad \frac{m^2\omega^2}{4\pi\hbar^2} \left(\frac{m\omega}{\hbar}\right)^N \frac{1}{k_1!k_2!k_3!k_4!} \left(\sqrt{r_i r_f}\cos\frac{\theta_i}{2}\cos\frac{\theta_f}{2}\right)^{k_1+k_3} \left(\sqrt{r_i r_f}\sin\frac{\theta_i}{2}\sin\frac{\theta_f}{2}\right)^{k_2+k_4} \\
&\quad \mathscr{P}\left(r_i\cos^2\frac{\theta_i}{2},\, r_i\sin^2\frac{\theta_i}{2}\right) \mathscr{P}\left(r_f\cos^2\frac{\theta_f}{2},\, r_f\sin^2\frac{\theta_f}{2}\right) e^{i(k_1-k_2-k_3+k_4)(\phi_i-\phi_f)/2}\,. \qquad (25)
\end{aligned}
$$

Similarly, we use τ_- and we have

$$
\begin{aligned}
\Psi_N(\boldsymbol{u}) =& \frac{m\omega}{\pi\hbar}\left(\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathscr{P}\left(\tilde{r}\sin^2\frac{\tilde{\theta}}{2},\, \tilde{r}\cos^2\frac{\tilde{\theta}}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} \left(\sqrt{\tilde{r}}\sin\frac{\tilde{\theta}}{2}\right)^{k_1+k_3} \left(\sqrt{\tilde{r}}\cos\frac{\tilde{\theta}}{2}\right)^{k_2+k_4} e^{-\frac{m\omega}{2\hbar}\tilde{r}} \\
&\exp\left[i(k_1+3k_2-k_3-3k_4)\frac{\tilde{\phi}}{2}\right] \exp\left[-i(k_1+k_2-k_3-k_4)\frac{\tilde{\nu}}{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
\Psi_{N,l}(\boldsymbol{u}) =& \frac{m\omega}{\pi\hbar}\left(\sqrt{\frac{m\omega}{\hbar}}\right)^N \frac{\mathscr{P}\left(\tilde{r}\sin^2\frac{\tilde{\theta}}{2},\, \tilde{r}\cos^2\frac{\tilde{\theta}}{2}\right)}{\sqrt{k_1!k_2!k_3!k_4!}} \left(\sqrt{\tilde{r}}\sin\frac{\tilde{\theta}}{2}\right)^{k_1+k_3} \left(\sqrt{\tilde{r}}\cos\frac{\tilde{\theta}}{2}\right)^{k_2+k_4} e^{-\frac{m\omega}{2\hbar}\tilde{r}} \\
&\exp\left[i(k_1+3k_2-k_3-3k_4)\frac{\tilde{\phi}}{2}\right] \exp\left[i\,l\,\frac{\tilde{\nu}}{2}\right],
\end{aligned}
$$

$$
\begin{aligned}
&G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E = -m\omega^2/8) \\
&= \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, \Psi_N(\boldsymbol{x}_f)\, \Psi_N^*(\boldsymbol{x}_i) \\
&= \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, e^{-\frac{m\omega}{2\hbar}(\tilde{r}_i + \tilde{r}_f)} \\
&\quad \frac{m^2\omega^2}{4\pi\hbar^2} \left(\frac{m\omega}{\hbar}\right)^N \frac{1}{k_1!k_2!k_3!k_4!} \left(\sqrt{\tilde{r}_i \tilde{r}_f}\sin\frac{\tilde{\theta}_i}{2}\sin\frac{\tilde{\theta}_f}{2}\right)^{k_1+k_3} \left(\sqrt{\tilde{r}_i \tilde{r}_f}\cos\frac{\tilde{\theta}_i}{2}\cos\frac{\tilde{\theta}_f}{2}\right)^{k_2+k_4} \\
&\quad \mathscr{P}\left(\tilde{r}_i\sin^2\frac{\tilde{\theta}_i}{2},\, \tilde{r}_i\cos^2\frac{\tilde{\theta}_i}{2}\right) \mathscr{P}\left(\tilde{r}_f\sin^2\frac{\tilde{\theta}_f}{2},\, \tilde{r}_f\cos^2\frac{\tilde{\theta}_f}{2}\right) e^{-i(k_1+3k_2-k_3-3k_4)(\tilde{\phi}_i-\tilde{\phi}_f)/2}\,. \qquad (26)
\end{aligned}
$$

By the other reduction process which is carried out in § 4, the same results as (25) and (26) are obtained from (17) as follows.

$$
\begin{aligned}
&G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E = -m\omega^2/8)\\
&= u_f^2 \lim_{\chi \to 4\pi - 0} \int_0^{\chi} G(\boldsymbol{u}_f,\, \boldsymbol{u}_i; 4k) \exp\left(i\, l\, \frac{\nu_i - \nu_f}{2}\right) d\nu_i\\
&= u_f^2 \lim_{\chi \to 4\pi - 0} \int_0^{\chi} \left\{ \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, \Psi_N(\boldsymbol{u}_f)\, \Psi_N^*(\boldsymbol{u}_i) \right\} \exp\left(i\, l\, \frac{\nu_i - \nu_f}{2}\right) d\nu_i\\
&= \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, e^{-\frac{m\omega}{2\hbar}(r_i + r_f)}\\
&\quad \frac{m^2\omega^2}{4\pi\hbar^2} \left(\frac{m\omega}{\hbar}\right)^N \frac{1}{k_1!k_2!k_3!k_4!} \left(\sqrt{r_i r_f} \cos\frac{\theta_i}{2} \cos\frac{\theta_f}{2}\right)^{k_1+k_3} \left(\sqrt{r_i r_f} \sin\frac{\theta_i}{2} \sin\frac{\theta_f}{2}\right)^{k_2+k_4}\\
&\quad \mathscr{P}\left(r_i \cos^2\frac{\theta_i}{2},\, r_i \sin^2\frac{\theta_i}{2}\right) \mathscr{P}\left(r_f \cos^2\frac{\theta_f}{2},\, r_f \sin^2\frac{\theta_f}{2}\right) e^{i(k_1 - k_2 - k_3 + k_4)(\phi_i - \phi_f)/2},
\end{aligned}
$$

$$
\begin{aligned}
&G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E = -m\omega^2/8)\\
&= u_f^2 \lim_{\chi \to 4\pi - 0} \int_0^{\chi} G(\boldsymbol{u}_f,\, \boldsymbol{u}_i; 4k) \exp\left(i\, l\, \frac{\tilde{\nu}_i - \tilde{\nu}_f}{2}\right) d\tilde{\nu}_i\\
&= u_f^2 \lim_{\chi \to 4\pi - 0} \int_0^{\chi} \left\{ \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, \Psi_N(\boldsymbol{u}_f)\, \Psi_N^*(\boldsymbol{u}_i) \right\} \exp\left(i\, l\, \frac{\tilde{\nu}_i - \tilde{\nu}_f}{2}\right) d\tilde{\nu}_i\\
&= \sum_{N=0}^{\infty} \frac{1}{4k - (N+2)\hbar\omega}\, e^{-\frac{m\omega}{2\hbar}(\tilde{r}_i + \tilde{r}_f)}\\
&\quad \frac{m^2\omega^2}{4\pi\hbar^2} \left(\frac{m\omega}{\hbar}\right)^N \frac{1}{k_1!k_2!k_3!k_4!} \left(\sqrt{\tilde{r}_i \tilde{r}_f} \sin\frac{\tilde{\theta}_i}{2} \sin\frac{\tilde{\theta}_f}{2}\right)^{k_1+k_3} \left(\sqrt{\tilde{r}_i \tilde{r}_f} \cos\frac{\tilde{\theta}_i}{2} \cos\frac{\tilde{\theta}_f}{2}\right)^{k_2+k_4}\\
&\quad \mathscr{P}\left(\tilde{r}_i \sin^2\frac{\tilde{\theta}_i}{2},\, \tilde{r}_i \cos^2\frac{\tilde{\theta}_i}{2}\right) \mathscr{P}\left(\tilde{r}_f \sin^2\frac{\tilde{\theta}_f}{2},\, \tilde{r}_f \cos^2\frac{\tilde{\theta}_f}{2}\right) e^{-i(k_1 + 3k_2 - k_3 - 3k_4)(\tilde{\phi}_i - \tilde{\phi}_f)/2}.
\end{aligned}
$$

This fact suggests that the reduction process executed in § 4 compares with the other process which is founded on the concept of restricting the Hilbert space $L^2(\mathbb{R}^4; 4u^2du)$ to the Hilbert space Γ_l of square integrable cross sections in the complex line bundles L_l over $\dot{\mathbb{R}}^3$.

Finally, Proposition 2 is rewrited as follows.

Proposition 3 (i) *When $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_+$, the Green's function of the MIC-Kepler problem is*

$$
\begin{aligned}
&G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E=-m\omega^2/8) \\
&= \sum_{N=0}^{\infty} \frac{1}{4k-(N+2)\hbar\omega}\, e^{-\frac{m\omega}{2\hbar}(r_i+r_f)}\, e^{i(k_1-k_2-k_3+k_4)(\phi_i-\phi_f)/2} \\
&\frac{m^2\omega^2}{4\pi\hbar^2}\left(\frac{m\omega}{\hbar}\right)^N \frac{1}{k_1!k_2!k_3!k_4!}\left(\sqrt{r_ir_f}\cos\frac{\theta_i}{2}\cos\frac{\theta_f}{2}\right)^{k_1+k_3}\left(\sqrt{r_ir_f}\sin\frac{\theta_i}{2}\sin\frac{\theta_f}{2}\right)^{k_2+k_4} \\
&\sum_{j=0}^{k_1}\sum_{s=0}^{k_2} j!s!\left(\frac{-\hbar}{m\omega}\right)^{j+s} {}_{k_1}C_j\cdot{}_{k_3}C_j\cdot{}_{k_2}C_s\cdot{}_{k_4}C_s\left(r_i\cos^2\frac{\theta_i}{2}\right)^{-j}\left(r_i\sin^2\frac{\theta_i}{2}\right)^{-s} \\
&\sum_{j'=0}^{k_1}\sum_{s'=0}^{k_2} j'!s'!\left(\frac{-\hbar}{m\omega}\right)^{j'+s'} {}_{k_1}C_{j'}\cdot{}_{k_3}C_{j'}\cdot{}_{k_2}C_{s'}\cdot{}_{k_4}C_{s'}\left(r_f\cos^2\frac{\theta_f}{2}\right)^{-j'}\left(r_f\sin^2\frac{\theta_f}{2}\right)^{-s'}.
\end{aligned}
$$

(ii) *When $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_-$, then the Green's function is written as*

$$
\begin{aligned}
&G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E=-m\omega^2/8) \\
&= \sum_{N=0}^{\infty} \frac{1}{4k-(N+2)\hbar\omega}\, e^{-\frac{m\omega}{2\hbar}(\tilde r_i+\tilde r_f)}\, e^{-i(k_1+3k_2-k_3-3k_4)(\tilde\phi_i-\tilde\phi_f)/2} \\
&\frac{m^2\omega^2}{4\pi\hbar^2}\left(\frac{m\omega}{\hbar}\right)^N \frac{1}{k_1!k_2!k_3!k_4!}\left(\sqrt{\tilde r_i\tilde r_f}\sin\frac{\tilde\theta_i}{2}\sin\frac{\tilde\theta_f}{2}\right)^{k_1+k_3}\left(\sqrt{\tilde r_i\tilde r_f}\cos\frac{\tilde\theta_i}{2}\cos\frac{\tilde\theta_f}{2}\right)^{k_2+k_4} \\
&\sum_{j=0}^{k_1}\sum_{s=0}^{k_2} j!s!\left(\frac{-\hbar}{m\omega}\right)^{j+s} {}_{k_1}C_j\cdot{}_{k_3}C_j\cdot{}_{k_2}C_s\cdot{}_{k_4}C_s\left(\tilde r_i\sin^2\frac{\tilde\theta_i}{2}\right)^{-j}\left(\tilde r_i\cos^2\frac{\tilde\theta_i}{2}\right)^{-s} \\
&\sum_{j'=0}^{k_1}\sum_{s'=0}^{k_2} j'!s'!\left(\frac{-\hbar}{m\omega}\right)^{j'+s'} {}_{k_1}C_{j'}\cdot{}_{k_3}C_{j'}\cdot{}_{k_2}C_{s'}\cdot{}_{k_4}C_{s'}\left(\tilde r_f\sin^2\frac{\tilde\theta_f}{2}\right)^{-j'}\left(\tilde r_f\cos^2\frac{\tilde\theta_f}{2}\right)^{-s'}.
\end{aligned}
$$

(iii) *When $\boldsymbol{x}_i$, $\boldsymbol{x}_f \in U_+ \cap U_-$, $G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E)$ is also written by $r=\tilde r$, $\theta=\pi-\tilde\theta$ and $\phi=\tilde\phi$ as*

$$G_-(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E) = e^{il(\phi_i-\phi_f)}\, G_+(\boldsymbol{x}_f,\, \boldsymbol{x}_i; E)\,.$$

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